

Tight lower bound for percolation threshold on a quasi-regular graph

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We construct an exact expression for the site percolation threshold p_c on a quasi-regular tree \mathcal{T} , and a related exact lower bound for a quasi-regular graph \mathcal{G} . Both are given by the inverse spectral radius of the appropriate Hashimoto matrix used to count non-backtracking walks. The obtained bound always exceeds the inverse spectral radius of the original graph, and it is also generally tighter than the existing bound in terms of the maximum degree.

An ability to process and store large amounts of information lead to emergence of big data in many areas of research and applications. This caused a renewed interest in graph theory as a tool for describing complex connections in various kinds of networks: social, biological, technological, etc.[1–5] In particular, percolation transition on graphs has been used to describe internet stability, spread of contagious diseases, and emergence of viral videos. Percolation has also been applied to establish the existence of the decoding threshold in certain classes of quantum error-correcting codes[6].

A *degree* of a vertex in a graph is the number of its neighbors. Degree distribution is a characteristic easy to extract empirically. A simple approach for network modeling is to study random graphs with the given degree distribution[7–9]. In the absence of correlations, the site percolation threshold on such a random graph is[7, 8]

$$p_c = \frac{\langle d \rangle}{\langle d^2 \rangle - \langle d \rangle}, \quad (1)$$

where $\langle d^m \rangle \equiv \sum_v d_v^m / n$ is the m -th moment of the vertex degree distribution and the *graph order*, n , is the number of vertices in the graph. While this result is very appealing in its simplicity, Eq. (1) has no predictive power for any actual network where correlations between degrees or enhanced connectivity (“clustering”) of nearby vertices may be present. Substantial effort has been spent on attempts to account for such correlations[10–12] in random graphs. However, such approaches can only account for local correlations and are flawed when applied to artificial networks like the power grid, which may have a carefully designed robust backbone (e.g., as in Example 1). Such correlations make Eq. (1) or its versions accounting for local correlations seemingly irrelevant.

There are only a handful of results on percolation for general graphs[13, 14]. These include the exact lower bound for the site percolation threshold for any graph with the maximum vertex degree d_{\max} [15],

$$p_c \geq (d_{\max} - 1)^{-1}, \quad (2)$$

which coincides with that for the bond percolation[14]. Both bounds are achieved on d -regular tree \mathcal{T}_d . Unfortunately, for graphs with wide degree distributions, Eq. (2) may easily underestimate the percolation threshold.

An estimate of the percolation threshold for dense graphs (with some conditions) as the inverse *spectral radius* of the graph, $\rho(\mathcal{G}) \equiv \rho(A_{\mathcal{G}})$, the largest eigenvalue of its adjacency matrix, $A_{\mathcal{G}}$, has been suggested in Ref. 16. Unfortunately, the conditions are rather restrictive, and the estimate is clearly not very accurate for sparse degree-regular graphs where the spectral radius $\rho(\mathcal{G}) = d$, as this estimate never reaches the lower bound in Eq. (2).

Example 1. Consider a tree graph $\mathcal{T} \equiv \mathcal{T}_{d;r,L}$ constructed by attaching r chains of length L to each vertex of a d -regular tree \mathcal{T}_d , see Fig. 1. The percolation threshold coincides with that of \mathcal{T}_d , $p_c = p_c(\mathcal{T}_d) = (d - 1)^{-1}$. On the other hand, Eq. (1) gives $p_c \rightarrow 0$ if we take $L = 1$, r large, and $p_c \rightarrow 1$ if we take $r = 1$, L large. Similarly, the spectral radius is $\rho(\mathcal{T}_{d;r,1}) = d/2 + [(d/2)^2 + r]^{1/2}$ (we took $L = 1$); the corresponding estimated threshold varies in the range $0 < [\rho(\mathcal{G})]^{-1} \leq 1/d$, while the lower bound (2) varies in the range $0 < p_c^{\min} \leq (d - 1)^{-1}$.

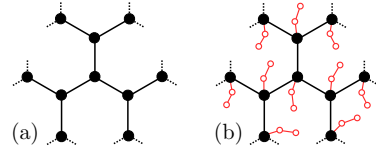


FIG. 1. (Color online) (a) A d -regular tree used for the backbone of the graph in Example 1. (b) The tree $\mathcal{T}_{d;r,L}$ is grown from the backbone by placing r chains of fixed length L (shown $d = 3$, $r = 1$, $L = 2$) at each vertex of the backbone.

Thus, Eq. (1), the lower bound (2), or the inverse spectral radius $[\rho(\mathcal{G})]^{-1}$ do not give accurate estimates of the percolation threshold for this graph family.

In this work we construct an exact expression for the percolation threshold on any quasi-transitive tree \mathcal{T} , and a related exact lower bound for the percolation threshold on a quasi-transitive graph \mathcal{G} which is more specific than Eq. (2). These are given by the inverse spectral radius of the *oriented line graph* (OLG) \mathcal{F} introduced by Kotani and Sunada[17]. The corresponding adjacency matrix, $A_{\mathcal{F}}$, is the Hashimoto matrix[18] used to enumerate non-backtracking walks on \mathcal{G} . We also show that the inverse spectral radius $\rho(\mathcal{G})$ of the original graph gives a smaller

(inexact) lower bound for the percolation threshold,

$$p_c \geq 1/\rho(\mathcal{F}) > 1/\rho(\mathcal{G}). \quad (3)$$

We call a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set \mathcal{V} and edge set \mathcal{E} *transitive* iff for any two vertices u, v in \mathcal{V} there is an automorphism of \mathcal{G} mapping u onto v . Graph \mathcal{G} is *quasi transitive* if there is a finite set of vertices $\mathcal{V}_0 \subset \mathcal{V}$ such that any $v \in \mathcal{V}$ is taken into \mathcal{V}_0 by some automorphism of \mathcal{G} . We say that any vertex which can be mapped onto a vertex $v_0 \in \mathcal{V}_0$ is in the equivalence class of v_0 .

In site percolation on a graph \mathcal{G} , each vertex is open with probability p and closed with probability $1 - p$; two neighboring open vertices belong to the same cluster. Percolation happens if there is an infinite cluster on \mathcal{G} .

First consider a quasi transitive *tree* \mathcal{T} , a graph with no cycles. According to Eq. (2), the corresponding percolation threshold must be strictly non-zero, $p_c \equiv p_c(\mathcal{T}) > 0$. Percolation threshold on any tree can be found exactly by constructing a set of recursive equations starting with some arbitrarily chosen root[19]. For a given open vertex i , let us introduce the probability Q_{ij} that i is connected to a finite cluster through its neighbor j . The corresponding recursive equations have the form

$$Q_{ij} = \prod_{l \sim j: l \neq i} (1 - p + pQ_{jl}), \quad (4)$$

where the product is taken over all neighbors l of j (denoted $l \sim j$) such that $l \neq i$ so that only so far uncovered independent branches are included. The growth of a branch into an infinite cluster is impeded by a neighboring site being closed (probability $1 - p$), or being open but connecting to a finite branch (probability pQ_{jl}).

Below the percolation threshold, $p < p_c$, Eqs. (4) are satisfied identically with $Q_{ij} = 1$. Right at the percolation threshold, we expect the probability of an infinite cluster to be vanishingly small, and the probabilities Q_{ij} can be expanded

$$Q_{ij} = 1 - \epsilon_{ij}, \quad i \sim j, \quad (5)$$

where ϵ_{ij} is infinitesimal. Expanding Eqs. (4) to linear order in ϵ_{ij} , we obtain the following eigenvalue problem at the threshold, $p = p_c$,

$$\lambda \epsilon_{ij} = \sum_{l \sim j: l \neq i} \epsilon_{jl}, \quad \lambda \equiv 1/p_c. \quad (6)$$

The percolation threshold corresponds to the largest eigenvalue λ corresponding to a non-negative eigenvector, $\epsilon_{ij} \geq 0$. To ensure the probability $p_c \leq 1$, the eigenvalue needs to be sufficiently large, $\lambda \geq 1$. It is convenient to extend Eqs. (6) to an arbitrary graph \mathcal{G} , where $\epsilon_{ij} \neq 0$ iff the corresponding component of the adjacency matrix is nonzero, $A_{ij} \neq 0$, including any diagonal elements, $i = j$, corresponding to loops in \mathcal{G} .

The eigenvalue problem (6) has a non-symmetric matrix with non-negative elements. According to Perron-Frobenius theory[20–22] of non-negative matrices, there always exists a non-negative solution with the eigenvalue λ equal to the spectral radius $\rho \geq 0$ of this matrix, although in general it is possible to have $\rho = 0$.

To establish a lower bound on ρ , we first construct a graphical interpretation of Eqs. (6). The components ϵ_{ij} correspond to directed edges of the original graph; the entire set corresponds to sites of the *line digraph* [23] associated with the symmetric digraph $\tilde{\mathcal{G}}$ equivalent to the original graph \mathcal{G} . Namely, each edge $(i, j) \in \mathcal{E}(\mathcal{G})$ is replaced by a pair of directed edges, $\{(i; j), (j; i)\} \subset \mathcal{E}(\tilde{\mathcal{G}})$. The summation over l in the r.h.s. of Eq. (6) would correspond to the adjacency matrix of the line digraph of $\tilde{\mathcal{G}}$, were it not for the exclusion $l \neq i$. With such a restriction, we obtain the adjacency matrix of the OLG [17] $\mathcal{F}_{\tilde{\mathcal{G}}}$ (technically, this is a digraph).

Generally, given a digraph $\mathcal{D} = (\mathcal{V}, \mathcal{E})$, the associated OLG $\mathcal{F}_{\mathcal{D}}$ has the vertex set $\mathcal{V}(\mathcal{F}_{\mathcal{D}}) = \mathcal{E}(\mathcal{D})$ and directed edges $((i; j); (j; l))$ such that $\{(i; j), (j; l)\} \subset \mathcal{E}(\mathcal{D})$ and $l \neq i$. To simplify notations, we will call the OLG $\mathcal{F}_{\mathcal{G}}$ of a graph \mathcal{G} the OLG $\mathcal{F}_{\tilde{\mathcal{G}}}$ of the corresponding symmetric digraph $\tilde{\mathcal{G}}$. By construction, Eqs. (6) are the eigenvalue equations for the adjacency matrix of the OLG, $A_{\mathcal{F}_{\mathcal{G}}}$. This matrix is also known as the Hashimoto matrix of the original graph[18]. We will prove the following

Theorem 1. *The largest real-valued eigenvalue λ of Eqs. (6) corresponding to a non-trivial eigenvector with non-negative components, $\epsilon_{ij} \geq 0$, is given by the spectral radius of the OLG, $\lambda_{\max} = \rho(\mathcal{F}_{\mathcal{G}})$. It satisfies $\lambda_{\max} \geq 1$ for any connected quasi transitive graph \mathcal{G} which is not a finite tree.*

Let Γ be a group of automorphisms of a graph \mathcal{G} . The *quotient graph* \mathcal{G}/Γ is the graph whose vertices are equivalence classes $\mathcal{V}(\mathcal{G})/\Gamma = \{\Gamma v : v \in \mathcal{V}(\mathcal{G})\}$, and an edge $(\Gamma u, \Gamma v)$ appears in \mathcal{G}/Γ if there are representatives $u_0 \in \Gamma u$ and $v_0 \in \Gamma v$ that are neighbors in \mathcal{G} , $(u_0, v_0) \in \mathcal{E}(\mathcal{G})$. Same definition applies in the case of a digraph \mathcal{D} , except that we need to consider directed edges, e.g., $(u_0; v_0) \in \mathcal{E}(\mathcal{D})$. Notice that a pair of equivalent neighboring vertices in the original (di)graph \mathcal{G} produces a loop in the quotient (di)graph \mathcal{G}/Γ . Notice also that an automorphism γ of the digraph \mathcal{D} induces a unique automorphism φ_γ of the corresponding OLG $\mathcal{F}_{\mathcal{D}}$, and a group Γ of automorphisms of \mathcal{D} induces an isomorphic group Φ_Γ of automorphisms of $\mathcal{F}_{\mathcal{D}}$. We will need the following two Lemmas:

Lemma 2. *A finite quotient graph $\mathcal{F}_0 \equiv \mathcal{F}_{\mathcal{G}}/\Phi_\Gamma$ of the OLG $\mathcal{F}_{\mathcal{G}}$ of any connected quasi-transitive graph \mathcal{G} with automorphism group Γ is strongly connected if the minimum and the maximum vertex degrees of \mathcal{G} satisfy $d_{\max}(\mathcal{G}) > 2$ and $d_{\min}(\mathcal{G}) > 1$.*

Proof. We are going to prove that for every ordered pair of vertices $(u; v)$ in \mathcal{F}_G there is a directed path between some vertices $u_0 \in \Phi_\Gamma v$ and $v_0 \in \Phi_\Gamma u$, respectively equivalent to u, v under the isomorphism group Φ_Γ induced by Γ . The vertices $\{u, v\} \subset \mathcal{F}_G$ are directed edges in the digraph $\tilde{\mathcal{G}}$; denote the corresponding undirected edges $\{e_u, e_v\} \subset \mathcal{E}(\mathcal{G})$. Connectivity of \mathcal{G} implies the existence of a path on \mathcal{G} connecting a vertex in e_u and a vertex in e_v which does not include these two edges. Thus, there is a directed path on $\tilde{\mathcal{G}}$ connecting either u or reverse of u with either v or reverse of v .

To ensure the existence of a directed path between the directed edges equivalent to actual u and v , we may just construct a directed non-backtracking path from any directed edge $u \in \mathcal{E}(\tilde{\mathcal{G}})$ to one of the edges $\bar{u}_0 \in \Phi_\Gamma \bar{u}$ equivalent to its reverse, \bar{u} . Since there are no degree-one vertices on \mathcal{G} , with a finite number of vertex equivalence classes induced by Γ , any non-backtracking path p starting with u will eventually come to a vertex equivalent to that already in the path; the corresponding path Γp on the quotient graph \mathcal{G}/Γ loops back onto itself. Consider two such paths p_1 and $p_2 \neq p_1$ starting at u ; they exist since there is at least one vertex with degree $d > 2$ in the graph \mathcal{G} . If either of Γp_1 or Γp_2 loops back onto itself on \mathcal{G}/Γ at a point other than the tail of u , we can complete a portion of that path in the reverse direction to arrive at some \bar{u}_0 equivalent to the reverse of u . Otherwise (both p_1 and p_2 end at equivalents of the tail of u), the required path on \mathcal{G}/Γ is Γp_1 joined with the reverse of Γp_2 , with any backtracking segments in the resulting path removed. In either case, the corresponding path on \mathcal{F}_G connects u with an equivalent of its reverse, \bar{u} ; its image under Φ_Γ is the path connecting $u \in \mathcal{F}_G/\Phi_\Gamma$ with the corresponding reverse, $\bar{u} \in \mathcal{F}_G/\Phi_\Gamma$. \square

Lemma 3. *A non-trivial solution of Eqs. (6) with an eigenvalue $\lambda \geq 1$ and a positive-component eigenvector satisfying the condition $\epsilon_{ij} = \epsilon_{i'j'}$ for any two ordered pairs of adjacent vertices $(i; j)$ and $(i'; j')$ that can be mapped onto each other by some automorphism of \mathcal{G} exists and is unique for any connected quasi-transitive graph \mathcal{G} with vertex degrees limited by $d_{\max}(\mathcal{G}) > 2$ and $d_{\min}(\mathcal{G}) > 1$.*

Proof. The ansatz leaves a finite eigensystem with a matrix M whose non-zero elements correspond to the adjacency matrix of the quotient graph of the OLG, $\mathcal{F}_0 \equiv \mathcal{F}_G/\Phi_\Gamma$. The statement of the Lemma follows from Lemma 2 and Perron-Frobenius theorem[20–22]. \square

Note that the eigenvalue in Lemma 3 is given by the spectral radius of M and is bounded from above and below by the spectral radii of \mathcal{F}_G and \mathcal{F}_0 , respectively:

$$1 \leq \rho(\mathcal{F}_G/\Phi_\Gamma) \leq \lambda = \rho(M) \leq \lambda_{\max} = \rho(\mathcal{F}_G). \quad (7)$$

Proof of Theorem 1. Define a *backbone* \mathcal{B} of a graph \mathcal{G} , a result of the recursive removal of all degree-one vertices.

For any finite tree the backbone is empty. For a connected graph \mathcal{G} which is not a finite tree, the backbone \mathcal{B} satisfies $d_{\min}(\mathcal{B}) > 1$. If \mathcal{G} is a connected quasi-transitive graph, so is \mathcal{B} . If, in addition, $d_{\max}(\mathcal{B}) > 2$, then the backbone \mathcal{B} satisfies the conditions of Lemma 3 which gives an explicit solution in this case. Otherwise, \mathcal{B} is a connected degree-regular graph with $d_{\max} = d_{\min} = 2$; it is a simple cycle or an infinite chain. In this case the adjacency matrix $A(\mathcal{F}_B/\Phi_\Gamma)$ has two independent strongly-connected components corresponding to the two classes of non-backtracking paths on \mathcal{B} ; the corresponding eigenvalue $\rho(\mathcal{F}_B/\Phi_\Gamma) = 1$ is doubly-degenerate. In either case, the only admissible eigenvalue is given by the spectral radius, $\lambda_{\max} \geq \lambda = \rho(M) \geq \rho(\mathcal{F}_B/\Phi_\Gamma) \geq 1$, see Eq. (7).

The original graph \mathcal{G} can be restored from the backbone \mathcal{B} by restoring degree-one vertices in the opposite order starting from the last removed. For such a vertex v which is connected to the vertex u already in the graph, we notice that $\epsilon_{uv} = 0$ (this bond cannot lead to an infinite cluster), while ϵ_{vu} is determined by the values ϵ_{uj} for bonds already in the graph, see Eq. (6), where the same eigenvalue λ must be used. Thus, additional vertices in \mathcal{F}_G cannot modify the components $\epsilon_{ij} \geq 0$ with $\{i, j\} \subset \mathcal{V}(\mathcal{B})$, and the maximum eigenvalue remains the same, $\lambda_{\max} = \rho(\mathcal{F}_G) = \rho(\mathcal{F}_B) \geq \rho(\mathcal{F}_B/\Phi_\Gamma) \geq 1$. \square

We next apply the constructed mean field theory to calculating percolation thresholds of more general graphs which may contain cycles. The main result of this work will be the following

Theorem 4. *The percolation threshold for any simple quasi-transitive graph \mathcal{G} which is not a finite tree is bounded from below by the inverse spectral radius of the corresponding OLG, $p_c(\mathcal{G}) \geq 1/\rho(\mathcal{F}_G)$.*

This bound is tight, as it becomes an equality for quasi-transitive trees. The approach is to construct a tree graph \mathcal{T} which is locally indistinguishable from the original graph \mathcal{G} , except that a closed walk on \mathcal{G} goes over to a walk connecting equivalent points on \mathcal{T} . We start by defining an operation for single cycle unwrapping (SCU) at a given bond which is not a bridge:

Definition 1. *Given a connected graph \mathcal{G} and a bond $b \equiv (u, v) \in \mathcal{E}(\mathcal{G})$, such that the two-terminal graph $\mathcal{G}' \equiv (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}) \setminus b)$ with source at v and sink at u is connected, define the cycle-unwrapped graph $\mathcal{C}_b \mathcal{G}$ as the series composition of an infinite chain of copies \mathcal{G}'_i , $i \in \mathbb{Z}$, of the graph \mathcal{G}' , with the source of \mathcal{G}'_i connected to the sink of the \mathcal{G}'_{i+1} .*

The SCU is illustrated in Fig. 2. Notice that for a graph with more than one cycle, unwrapping at b removes one cycle but creates an infinite number of copies of the remaining cycles. Nevertheless, for a locally finite graph, a countable number of SCUs is needed to remove all cycles. Indeed, the cycle-unwrapped image of

any path on \mathcal{G} that does not include b will remain entirely within a single copy of \mathcal{G}' . Thus, if at each SCU step we choose a bond b at distance r_b from some fixed origin vertex, such that only bridge bonds can be found closer to the origin, $r < r_b$, any copy of the remaining non-bridge bond introduced by the SCU is going to be at a distance $r > r_b$. Thus, each SCU reduces the number of non-bridge bonds at r_b , and for a locally finite graph, a finite number of SCUs is required to ensure that all bonds at $r = 0, 1, 2, \dots$ are bridge bonds. This proves

Lemma 5. *For a locally finite graph $\mathcal{G}_0 \equiv \mathcal{G}$, a sequence of SCUs $\mathcal{G}_{m+1} \equiv \mathcal{C}_{b_{m+1}}\mathcal{G}_m$ can be chosen so that in the $m \rightarrow \infty$ limit the resulting graph is a tree, $\mathcal{T} \equiv \mathcal{C}_\infty\mathcal{G}$.*

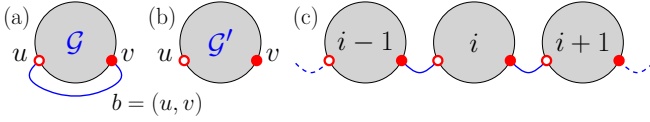


FIG. 2. (Color online) Illustration of SCU: (a) A graph \mathcal{G} with a non-bridge bond $b \equiv (u, v)$ highlighted; (b) Two-terminal graph \mathcal{G}' ; (c) The resulting graph $\mathcal{C}_b\mathcal{G}$ is a series composition of an infinite chain of copies of \mathcal{G}' .

Clearly, the graph $\mathcal{C}_b\mathcal{G}$ produced by an SCU has a group of isomorphisms \mathbb{Z} generated by the translation $i \rightarrow i + 1$. The corresponding graph quotient recovers the original graph, $\mathcal{G} = (\mathcal{C}_b\mathcal{G})/\mathbb{Z}$. This symmetry allows us to prove the following

Lemma 6. *For any simple quasi-transitive graph \mathcal{G} , SCU does not change the spectral radius of \mathcal{G} , $\rho(\mathcal{G}) = \rho(\mathcal{C}_b\mathcal{G})$, or of the OLG, $\rho(\mathcal{F}_\mathcal{G}) = \rho(\mathcal{F}_{\mathcal{C}_b\mathcal{G}})$.*

Proof. The symmetry of $\mathcal{C}_b\mathcal{G}$ implies that an eigenvector e can always be chosen to diagonalize both its adjacency matrix $A \equiv A(\mathcal{C}_b\mathcal{G})$, $Ae = \lambda e$, and the translation generator T , $Te = \mu e$. Translation group being Abelian, its representations are all one-dimensional, with $\mu = e^{ik}$, with $0 \leq k < 2\pi$. Let e_0 with non-negative components be the Perron-Frobenius eigenvector of the non-negative matrix A with the eigenvalue equal to its spectral radius, $\lambda_{\max} = \rho(A)$. Symmetrizing e_0 over \mathbb{Z} , gives a non-negative eigenvector e corresponding to the same λ_{\max} and $k = 0$. This corresponds to the ansatz introduced in Lemma 3, thus $\rho(M') = \rho(\mathcal{C}_b\mathcal{G})$, where M' is the reduced matrix corresponding to the symmetric eigenvector of A , cf. Eq. (7). Further, for a simple (di)graph \mathcal{G} , the matrix elements of M' satisfy $M'_{ij} \in \{0, 1\}$, thus $M' = A_\mathcal{G}$, which gives $\rho(\mathcal{G}) = \rho(M') = \rho(\mathcal{C}_b\mathcal{G})$. The proof in the case of $\mathcal{F}_{\mathcal{C}_b\mathcal{G}}$ is identical if we notice $\mathcal{C}_b\mathcal{F}_\mathcal{G} = \mathcal{F}_{\mathcal{C}_b\mathcal{G}}$. \square

Proof of Theorem 4. Vertex transitivity of \mathcal{G} implies that a finite maximum degree exists; according to Lemma 5 \mathcal{G} can be transformed to a tree \mathcal{T} by a series of SCUs. Each step of the sequence can be undone by a graph quotient, $\mathcal{G}_m = \mathcal{G}_{m+1}/\mathbb{Z}$. According to Theorem 1 in Ref. 13, the

percolation threshold of a graph quotient cannot be below that of the original graph, thus $p_c(\mathcal{G}_m) = p_c(\mathcal{G}_{m+1}/\mathbb{Z}) \geq p_c(\mathcal{G}_{m+1})$; the entire sequence gives $p_c(\mathcal{G}) \geq p_c(\mathcal{T})$. On the other hand, Eq. (6) and Theorem 1 give $p_c(\mathcal{T}) = 1/\rho(\mathcal{F}_\mathcal{T})$. Moreover, each of the intermediate graphs of the sequence is vertex transitive and simple, thus the spectral radius of corresponding OLG is preserved at each step, $\rho(\mathcal{F}_\mathcal{T}) = \rho(\mathcal{F}_\mathcal{G})$, see Lemma 6. \square

Finally, we establish the relation between the spectral radius of OLG with that of the original graph:

Theorem 7. *The spectral radius of any connected non-empty graph \mathcal{G} is strictly larger than that of the corresponding OLG, $\rho(\mathcal{G}) > \rho(\mathcal{F}_\mathcal{G})$.*

Proof. A non-empty graph contains at least one edge (or a loop), thus $\rho(\mathcal{G}) > 0$; we only need to consider the case where $\rho(\mathcal{F}_\mathcal{G}) > 0$. Begin with Eq. (6) and assume $\epsilon_{ij} \geq 0$ is the non-zero eigenvector corresponding to the eigenvalue $\lambda \equiv \rho(\mathcal{F}_\mathcal{G}) > 0$. Introduce vertex variables

$$y_i \equiv \sum_{j: j \sim i} \epsilon_{ij}, \quad (8)$$

corresponding to the sum of ϵ_{ij} over all directed bonds leaving a given vertex i . These variables satisfy[24]

$$[\lambda^2 I + (D - I)] y = \lambda A y, \quad (9)$$

where $D \equiv \text{diag}(d_1, \dots, d_n)$ is the diagonal matrix of degrees, I is the identity matrix, and $A \equiv A_\mathcal{G}$ is the (symmetric) adjacency matrix of \mathcal{G} . If we multiply Eq. (9) by y^T on the left, the r.h.s. does not exceed $\lambda \rho(A) \|y\|^2$. From the proof of Theorem 1 it follows that there exists a vertex on the backbone of \mathcal{G} with $d_v > 1$ such that the corresponding component of y is non-zero, thus $y^T(D - I)y > 0$; dropping this term gives $\lambda < \rho(A)$. \square

Corollary 8. *The percolation threshold for any quasi-transitive graph that is not a finite tree satisfies Eq. (3).*

In conclusion, we constructed an exact expression for the threshold of site percolation on an arbitrary quasi-transitive tree, and an associated exact lower bound for such a threshold on an arbitrary graph. These are given by the inverse spectral radius of the oriented line graph associated with the tree or the graph, respectively. The constructed bound accounts for local structure of the graph, and is asymptotically exact for graphs with no short loops. For degree-regular graphs it goes over into the known lower bound (2). We also demonstrated that the inverse spectral radius of the original graph \mathcal{G} which was suggested previously as an estimate for the percolation threshold is always strictly smaller than our lower bound, see Eq. (3). In applications, spectral radius for sparse graphs involving billions of edges can be readily evaluated using standard numerical packages.

Our results can be easily extended to the cases of Bernoulli (bond), combined site-bond, or non-uniform percolation, where the probabilities to have an open vertex may differ from site to site. A similar technique can also be used to prove the conjecture on the location of the threshold for vertex-dependent percolation on directed graphs[25].

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